CHAPTER 12: POLYNOMIALS

Complete Detailed Mathematics Notes

1. Introduction to Polynomials

A polynomial is an algebraic expression consisting of variables, constants, and non-negative integral powers of variables, combined using addition, subtraction, and multiplication.

A general polynomial in x is written as:

$$P(x)=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$$

where:

- $a_n, a_{n-1}, \cdots, a_0$ are constants called coefficients.
- $a_n \neq 0$.
- n is a non-negative integer.
- x is the variable.
- n is called the degree of the polynomial.

Examples

$$3x^2+5x-7$$

is a polynomial.

$$7x^4-3x^2+8$$

is a polynomial.

$$5$$

is also a polynomial.

Not Polynomials

- $1/x+2$ because x has power −1.
- $\sqrt{x}+3$ because x has power ½.
- $x^{-2}+x$ because negative powers are not allowed.

2. Important Parts of a Polynomial

Consider:

$$P(x)=5x^4-3x^3+7x^2-9x+12$$

The terms are:

$$5x^4,\ -3x^3,\ 7x^2,\ -9x,\ 12$$

Coefficients

The coefficients are:

$$5, -3, 7, -9, 12$$

Constant Term

The term independent of x is:

$$12$$

Leading Coefficient

The coefficient of the highest power of x is:

$$5$$

Degree

The highest power of x is 4.

Therefore:

$$\deg P(x) = 4$$

3. Terms of a Polynomial

A polynomial is divided into terms by + and − signs.

For:

$$P(x) = 4x^3 - 7x^2 + 5x - 9$$

there are four terms:

$$4x^3,\ -7x^2,\ 5x,\ -9$$

A polynomial may be:

- Monomial
- Binomial
- Trinomial
- Multinomial

4. Monomial

A polynomial containing only one term is called a monomial.

Examples

$$5x$$

$$7x^3$$

$$-9$$

$$12x^5$$

5. Binomial

A polynomial containing two terms is called a binomial.

Examples

$$x+5$$
$$x^2-9$$
$$3x^3+7x$$

6. Trinomial

A polynomial containing three terms is called a trinomial.

Examples

$$x^2+5x+6$$
$$3x^2-7x+2$$

7. Polynomial Degree Classification

8. Constant Polynomial

A non-zero constant polynomial has degree 0.

Example:

$$P(x)=7$$

Therefore:

$$\deg P=0$$

Special Case: Zero Polynomial

The zero polynomial is:

$$P(x)=0$$

Its degree is generally considered undefined.

9. Linear Polynomial

A polynomial of degree 1 is called a linear polynomial.

General form:

$$P(x)=ax+b,\ a\neq 0$$

Example:

$$3x-7$$

Its zero is:

$$x=7/3$$

10. Quadratic Polynomial

A polynomial of degree 2 is called a quadratic polynomial.

General form:

$$P(x)=ax^2+bx+c,\ a\neq 0$$

Example:

$$2x^2-5x+3$$

Important concepts related to quadratic polynomials include:

- Roots
- Zeros
- Factorisation
- Discriminant
- Sum of roots
- Product of roots

11. Cubic Polynomial

A polynomial of degree 3 is called a cubic polynomial.

General form:

$$P(x)=ax^3+bx^2+cx+d$$

where:

$$a\neq 0$$

Example:

$$2x^3-5x^2+3x-7$$

A cubic polynomial can have up to three real roots.

12. Degree of a Polynomial

The degree is the highest power of the variable having a non-zero coefficient.

Example

$$P(x)=7x^5-3x^2+9$$

Degree:

Example

$$P(x)=4x^3+8x^3-2x+1$$

First combine like terms:

$$P(x)=12x^3-2x+1$$

Therefore degree = 3.

13. Polynomial in More Than One Variable

A polynomial may contain several variables.

Example:

$$P(x,y)=3x^2y+5xy^2-7$$

The degree of each term is:

$$3x^2y \rightarrow 2+1=3$$
$$5xy^2 \rightarrow 1+2=3$$
$$-7 \rightarrow 0$$

Therefore the degree of the polynomial is:

$$3$$

14. Evaluation of a Polynomial

To find P(a), substitute:

$$x=a$$

in the polynomial.

Example

Given:

$$P(x)=2x^3-5x+7$$

Find P(2).

Solution

$$P(2)=2(2)^3-5(2)+7$$
$$=16-10+7$$
$$13$$

15. Polynomial Function

A polynomial can also define a function:

$$f(x)=a_n x^n+\cdots+a_1 x+a_0$$

For every value of x, the polynomial gives a corresponding value of f(x).

For example:

$$f(x)=x^2-4x+3$$

Then:

$$f(1)=1-4+3=0$$

16. Zero or Root of a Polynomial

A number a is called a zero or root of P(x) if:

$$P(a)=0$$

Example

$$P(x)=x^2-5x+6$$

Factorise:

$$P(x)=(x-2)(x-3)$$

Therefore:

$$P(2)=0$$

and:

$$P(3)=0$$

Hence 2 and 3 are zeros of the polynomial.

17. Geometrical Meaning of a Zero

For a polynomial function:

$$y=P(x)$$

the zeros are the points where the graph intersects or touches the x-axis.

If:

$$P(a)=0$$

then:

$$(a,0)$$

lies on the graph.

18. Number of Zeros

A polynomial of degree n can have at most n real zeros.

Examples:

- Linear — Maximum zeros: 1
- Quadratic — Maximum zeros: 2
- Cubic — Maximum zeros: 3
- Quartic — Maximum zeros: 4

19. Remainder Theorem

One of the most important results in polynomial problems is the Remainder Theorem.

If P(x) is divided by:

$$x - a$$

then the remainder is:

$$P(a)$$

Example

Find the remainder when:

$$P(x) = 2x^3 + 3x^2 - 5x + 7$$

is divided by:

$$x - 2$$

Solution

According to the Remainder Theorem:

$$\text{Remainder} = P(2)$$

$$= 2(8) + 3(4) - 10 + 7$$

$$= 16 + 12 - 10 + 7$$

$$25$$

20. Remainder When Divided by x+a

If the divisor is:

$$x + a$$

write:

$$x + a = x - (-a)$$

Therefore:

$$\text{Remainder}=P(-a)$$

Example

Find the remainder when:

$$P(x)=x^3+2x^2-5x+4$$

is divided by:

$$x+2$$

Then:

$$\text{Remainder}=P(-2)$$
$$=(-8)+8+10+4$$
$$14$$

21. Factor Theorem

If:

$$P(a)=0$$

then:

$$x-a$$

is a factor of $P(x)$.

Conversely, if $x-a$ is a factor of $P(x)$, then:

$$P(a)=0$$

This is called the Factor Theorem.

22. Factor Theorem Example

Determine whether $x-3$ is a factor of:

$$P(x)=x^3-4x^2+x+6$$

Solution

Calculate:

$$P(3)=27-36+3+6$$
$$=0$$

Therefore:

$$x-3 \text{ is a factor}$$

23. Finding an Unknown Constant Using Factor Theorem

Suppose:

$$P(x)=x^3-4x^2+kx+6$$

and $x-2$ is a factor.

Therefore:

$$P(2)=0$$
$$8-16+2k+6=0$$
$$2k-2=0$$
$$k=1$$

24. Polynomial Division

If P(x) is divided by D(x), then:

$$P(x)=D(x)Q(x)+R(x)$$

where:

- P(x) = dividend
- D(x) = divisor
- Q(x) = quotient
- R(x) = remainder

The degree of the remainder must be less than the degree of the divisor.

25. Important Polynomial Division Result

If the divisor has degree 1:

$$x-a$$

the remainder is a constant.

If the divisor has degree 2:

$$x^2+bx+c$$

the remainder can be at most linear:

$$px+q$$

26. Algebraic Identities

Polynomial problems frequently depend upon standard algebraic identities.

Identity 1

$$(a+b)^2=a^2+2ab+b^2$$

Identity 2

$$(a-b)^2=a^2-2ab+b^2$$

Identity 3

$$a^2-b^2=(a-b)(a+b)$$

27. Cubic Identities

Sum of Cubes

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Difference of Cubes

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

28. Cube of a Sum

$$(a+b)^3=a^3+b^3+3ab(a+b)$$

Expanded:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

29. Cube of a Difference

$$(a-b)^3=a^3-b^3-3ab(a-b)$$

Expanded:

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

30. Important Identity

$$a^3+b^3+c^3-3abc$$

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

A particularly important special case is:

If:

$$a+b+c=0$$

then:

$$a^3+b^3+c^3=3abc$$

31. Factorisation

Factorisation means expressing a polynomial as a product of simpler factors.

Example

$$x^2 - 9$$

Using:

$$a^2 - b^2 = (a - b)(a + b)$$

we get:

$$x^2 - 9 = (x - 3)(x + 3)$$

32. Factorisation by Common Factor

Consider:

$$6x^3 + 9x^2$$

Common factor:

$$3x^2$$

Therefore:

$$6x^3 + 9x^2 = 3x^2(2x + 3)$$

33. Factorisation of Quadratic Polynomial

For:

$$x^2 + 5x + 6$$

we need two numbers whose:

- product = 6
- sum = 5

They are 2 and 3.

Therefore:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

34. Factorisation of Cubic Polynomial

Consider:

$$x^3 - 6x^2 + 11x - 6$$

Try possible roots:

$$x=1$$

$$P(1)=1-6+11-6=0$$

Therefore:

$$x-1$$

is a factor.

Dividing gives:

$$x^2-5x+6$$

which factors as:

$$(x-2)(x-3)$$

Therefore:

$$x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$$

35. Relationship Between Roots and Factors

If:

$$P(x)=a(x-\alpha)(x-\beta)$$

then the roots are:

$$\alpha,\beta$$

The factor corresponding to root α is:

$$x-\alpha$$

36. Quadratic Polynomial and Roots

For:

$$ax^2+bx+c=0$$

with roots α and β:

$$\alpha+\beta=-b/a$$

$$\alpha\beta=c/a$$

These are called relations between roots and coefficients.

37. Forming a Quadratic Polynomial

If roots are α and β, then:

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Example

Roots are 3 and 5.

$$\alpha+\beta=8$$
$$\alpha\beta=15$$

Therefore:

$$x^2-8x+15=0$$

38. Polynomial with Given Sum and Product of Roots

Suppose:

$$\alpha+\beta=7$$

and:

$$\alpha\beta=10$$

Then:

$$x^2-7x+10$$

is the required polynomial.

39. Transformation of Roots

Suppose roots of:

$$ax^2+bx+c=0$$

are α,β.

We can find expressions involving the roots using:

$$\alpha+\beta=-b/a$$
$$\alpha\beta=c/a$$

Example

Find:

$$\alpha^2+\beta^2$$

Using:

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

Therefore:

$$\alpha^2+\beta^2=(-b/a)^2-2c/a$$

40. Important Root Expressions

If roots are α,β:

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

$$1/\alpha+1/\beta=(\alpha+\beta)/(\alpha\beta), \text{ provided } \alpha\beta\neq 0.$$

$$\alpha/\beta+\beta/\alpha=(\alpha^2+\beta^2)/(\alpha\beta)$$

41. Cubic Polynomial and Roots

For:

$$ax^3+bx^2+cx+d=0$$

with roots α,β,γ:

$$\alpha+\beta+\gamma=-b/a$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=c/a$$

$$\alpha\beta\gamma=-d/a$$

These are extremely useful for higher-degree polynomial problems.

42. Forming a Cubic Polynomial

If roots are:

$$\alpha,\beta,\gamma$$

then the polynomial is:

$$(x-\alpha)(x-\beta)(x-\gamma)$$

Expanding:

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$$

43. Example — Cubic Polynomial from Roots

Find the polynomial whose roots are:

$$1,2,4$$

Solution

$$P(x)=(x-1)(x-2)(x-4)$$

First:

$$(x-1)(x-2)=x^2-3x+2$$

Therefore:

$$P(x)=(x^2-3x+2)(x-4)$$
$$=x^3-7x^2+14x-8$$

Hence:

$$x^3-7x^2+14x-8$$

44. Multiplicity of a Root

If:

$$P(x)=(x-a)^2(x-b)$$

then a is a root of multiplicity 2.

For example:

$$P(x)=(x-2)^3(x+1)$$

has:

- root 2 with multiplicity 3
- root −1 with multiplicity 1

The total number of roots counted with multiplicity is equal to the degree.

45. Repeated Roots

For a quadratic:

$$ax^2+bx+c$$

if:

$$b^2-4ac=0$$

then the polynomial has equal roots.

The repeated root is:

$$-b/(2a)$$

46. Sign of a Polynomial

The sign of a polynomial can be determined using its factors.

Consider:

$$P(x)=(x-2)(x-5)$$

Critical points are:

$$x=2,\ x=5$$

Intervals:

$$(-\infty,2),\ (2,5),\ (5,\infty)$$

Test the sign in each interval.

This technique becomes especially useful while solving polynomial inequalities.

47. Polynomial Inequalities

Examples include:

$$x^2-5x+6>0$$

Factor:

$$(x-2)(x-3)>0$$

The expression is positive outside the roots.

Therefore:

$$x<2 \text{ or } x>3$$

48. Graphical Behaviour of Polynomials

For:

$$y=P(x)$$

the degree determines the general shape of the graph.

Even-degree polynomial

The two ends generally move in the same direction.

Odd-degree polynomial

The two ends generally move in opposite directions.

The leading coefficient determines whether the graph rises or falls at the ends.

49. Important Properties of Polynomial Graphs

For a polynomial:

$$P(x)$$

- It is continuous for all real x.
- It has no breaks or jumps.
- Its real zeros correspond to x-axis intersections.
- A root of even multiplicity generally causes the graph to touch and turn.

- A root of odd multiplicity generally causes the graph to cross the x-axis.

50. Rational Root Test

For a polynomial with integer coefficients:

$$a_n x^n + \cdots + a_0$$

if a rational number:

$$p/q$$

in lowest terms is a root, then:

- p divides the constant term a_0
- q divides the leading coefficient a_n

This provides a systematic way to identify possible rational roots.

51. Integer Root Problems

For a monic polynomial:

$$x^3 + ax^2 + bx + c$$

any integer root must be a factor of c.

Example

For:

$$x^3 - 6x^2 + 11x - 6$$

possible integer roots are factors of 6:

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Testing these helps identify the roots quickly.

52. Sum of Polynomial Values

Polynomial identities can simplify expressions such as:

$$P(a) + P(b)$$

or:

$$P(a) - P(b)$$

especially when a+b, ab, or other symmetric expressions are known.

Example

If:

$$P(x)=x^2-3x+2$$

then:

$$P(a)+P(b)=a^2+b^2-3(a+b)+4$$

If a+b and ab are known, use:

$$a^2+b^2=(a+b)^2-2ab$$

53. Polynomial Identity

Two polynomial expressions are identical if their coefficients of corresponding powers are equal.

For example:

$$(ax+b)(x+2)$$

and:

$$ax^2+(2a+b)x+2b$$

are identical.

This method is useful when an unknown coefficient must be determined.

54. Comparing Coefficients

Suppose:

$$(x+a)(x+b)=x^2+7x+12$$

Expanding:

$$x^2+(a+b)x+ab$$

Comparing coefficients:

$$a+b=7$$

$$ab=12$$

Therefore:

$$a=3,\ b=4$$

55. Polynomial with Divisibility Conditions

If:

$$P(x)$$

is divisible by both:

$$x-a$$

and:

$$x-b$$

where $a \neq b$, then:

$$P(a)=0$$

and:

$$P(b)=0$$

Therefore:

$$(x-a)(x-b)$$

is a factor of P(x).

56. Multiple Linear Factors

If a polynomial has roots:

$$a,b,c$$

then:

$$(x-a)(x-b)(x-c)$$

is a factor.

Example

If:

$$P(1)=P(2)=P(3)=0$$

then:

$$(x-1)(x-2)(x-3)$$

divides P(x).

57. Important Solved Example

Question

If $x-2$ and $x+3$ are factors of:

$$P(x)=x^3+ax^2+bx-6$$

find a and b.

Solution

Since $x-2$ is a factor:

$$P(2)=0$$
$$8+4a+2b-6=0$$
$$2a+b+1=0$$
$$2a+b=-1$$

Since x+3 is a factor:

$$P(-3)=0$$
$$-27+9a-3b-6=0$$
$$9a-3b=33$$
$$3a-b=11$$

Now:

$$2a+b=-1$$
$$3a-b=11$$

Adding:

$$5a=10$$
$$a=2$$

Then:

$$4+b=-1$$
$$b=-5$$

Therefore:

$$a=2,\ b=-5$$

58. Important Solved Example — Remainder

Question

Find the remainder when:

$$2x^4-3x^3+5x^2-7x+10$$

is divided by:

$$x+1$$

Solution

Use:

$$\text{Remainder}=P(-1)$$
$$=2(-1)^4-3(-1)^3+5(-1)^2-7(-1)+10$$

$$=2+3+5+7+10$$

$$27$$

59. Important Solved Example — Factorisation

Question

Factorise:

$$x^3-4x^2-x+4$$

Solution

Group terms:

$$x^2(x-4)-1(x-4)$$

Therefore:

$$(x^2-1)(x-4)$$

Using:

$$x^2-1=(x-1)(x+1)$$

Hence:

$$(x-1)(x+1)(x-4)$$

60. Important Solved Example — Root Relationship

Question

The roots of:

$$2x^2-7x+3=0$$

are α,β. Find:

$$\alpha^2+\beta^2$$

Solution

$$\alpha+\beta=-(-7)/2=7/2$$

and:

$$\alpha\beta=3/2$$

Therefore:

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$=49/4-3$$

$$=(49-12)/4$$

61. Important Solved Example — Cubic Roots

Question

If α,β,γ are roots of:

$$2x^3-3x^2+5x-7=0$$

find:

$$\alpha+\beta+\gamma$$

Solution

For:

$$ax^3+bx^2+cx+d=0$$

we have:

$$\alpha+\beta+\gamma=-b/a$$

Therefore:

$$-(-3)/2$$
$$3/2$$

62. Important Solved Example — Unknown Constant

Question

If $x+1$ is a factor of:

$$P(x)=2x^3+kx^2-5x+3$$

find k.

Solution

Since $x+1$ is a factor:

$$P(-1)=0$$
$$2(-1)^3+k(-1)^2-5(-1)+3=0$$
$$-2+k+5+3=0$$
$$k+6=0$$

Therefore:

$$k=-6$$

63. Important Solved Example — Polynomial Formation

Question

Form the monic polynomial whose roots are:

$$2, -3, 5$$

Solution

Required polynomial:

$$P(x) = (x-2)(x+3)(x-5)$$

First:

$$(x-2)(x+3) = x^2 + x - 6$$

Now:

$$P(x) = (x^2 + x - 6)(x-5)$$
$$= x^3 - 4x^2 - 11x + 30$$

Therefore:

$$x^3 - 4x^2 - 11x + 30$$

64. Important Solved Example — Remainder Condition

Question

When $P(x)$ is divided by $x-2$, the remainder is 5. What is $P(2)$?

Solution

By the Remainder Theorem:

$$P(2) = \text{remainder}$$

Therefore:

$$P(2) = 5$$

65. Important Solved Example — Two Divisors

Question

A polynomial $P(x)$ leaves remainder 4 when divided by $x-2$, and remainder 7 when divided by $x+1$. Find $P(2)$ and $P(-1)$.

Solution

Directly from the Remainder Theorem:

$$P(2)=4$$

and:

$$P(-1)=7$$

Therefore:

$$P(2)=4,\ P(-1)=7$$

66. Important Shortcuts

Shortcut 1

For divisor:

$$x-a$$

immediately substitute:

$$x=a$$

Shortcut 2

For divisor:

$$x+a$$

substitute:

$$-a$$

Shortcut 3

If:

$$P(a)=0$$

immediately conclude:

$$x-a$$

is a factor.

Shortcut 4

For quadratic roots:

$$\alpha+\beta=-b/a$$
$$\alpha\beta=c/a$$

Shortcut 5

For cubic roots:

$$\alpha+\beta+\gamma=-b/a$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=c/a$$
$$\alpha\beta\gamma=-d/a$$

67. Common Conceptual Traps

Trap 1

$$x^2+1/x$$

is not a polynomial in x.

Trap 2

The degree is determined by the highest power after simplification.

Trap 3

For x+a, substitute:

$$-a$$

not a.

Trap 4

A zero of P(x) is a value of x, not the entire factor.

If:

$$P(3)=0$$

then:

$$3$$

is the zero, while:

$$x-3$$

is the factor.

Trap 5

The remainder must have degree less than the divisor.

68. QUICK REVISION SHEET

Polynomial

$$P(x)=a_n x^n+\cdots+a_0$$

Degree

Highest power of the variable.

Zero

$$P(a)=0$$

Remainder Theorem

$$\text{Remainder on division by } x-a=P(a)$$

Factor Theorem

$$P(a)=0 \Leftrightarrow x-a \text{ is a factor}$$

Quadratic

$$ax^2+bx+c$$

Quadratic Root Relations

$$\alpha+\beta=-b/a$$
$$\alpha\beta=c/a$$

Cubic Root Relations

$$\alpha+\beta+\gamma=-b/a$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=c/a$$
$$\alpha\beta\gamma=-d/a$$

69. COMPLETE FORMULA & IDENTITY SHEET

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$
$$a^2-b^2=(a-b)(a+b)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$(a+b)^3=a^3+b^3+3ab(a+b)$$
$$(a-b)^3=a^3-b^3-3ab(a-b)$$
$$a^3+b^3+c^3-3abc$$
$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

If:

$$a+b+c=0$$

then:

70. FINAL SUMMARY

A strong understanding of polynomials requires mastery of:

- Definition and classification of polynomials
- Degree and coefficients
- Evaluation of polynomial functions
- Zeros and roots
- Remainder Theorem
- Factor Theorem
- Polynomial division
- Standard algebraic identities
- Factorisation
- Quadratic root relationships
- Cubic root relationships
- Formation of polynomials from roots
- Repeated roots and multiplicity
- Comparing coefficients
- Unknown coefficient problems
- Polynomial divisibility
- Rational and integer root techniques
- Polynomial inequalities
- Graphical interpretation
- Advanced applications of roots and coefficients

Degree	Name	Example
0	Constant	7
1	Linear	$3x+5$
2	Quadratic	x^2+4x+3
3	Cubic	x^3-2x+1
4	Quartic	x^4+3x^2+1
5	Quintic	x^5-x+2

The most important tools to remember are the Remainder Theorem, Factor Theorem, algebraic identities, and relationships between roots and coefficients.