Chapter 4: Trigonometry

Trigonometric Identities — Detailed Notes & Solved Examples

1. Introduction

A trigonometric identity is an equation involving trigonometric ratios that is true for every value of the angle for which both sides are defined.

Identities are used to simplify trigonometric expressions, prove relationships, and find unknown trigonometric ratios.

The most important identities are derived from the fundamental relationship between the sides of a right-angled triangle.

2. Fundamental Trigonometric Ratios

$$\sin\theta = P/H \qquad \cos\theta = B/H \qquad \tan\theta = P/B$$

$$\text{cosec}\,\theta = H/P \qquad \sec\theta = H/B \qquad \cot\theta = B/P$$

3. Reciprocal Identities

Each of the following pairs is reciprocal:

$$\sin\theta \times \text{cosec}\,\theta = 1$$

$$\cos\theta \times \sec\theta = 1$$

$$\tan\theta \times \cot\theta = 1$$

Therefore:

$$\text{cosec}\,\theta = 1/\sin\theta \qquad \sec\theta = 1/\cos\theta \qquad \cot\theta = 1/\tan\theta$$

4. Quotient Identities

$$\tan\theta = \sin\theta / \cos\theta$$

$$\cot\theta = \cos\theta / \sin\theta$$

These identities are useful when an expression contains sine and cosine but the required answer is in terms of tangent or cotangent.

5. Fundamental Pythagorean Identity

$$\sin^2\theta + \cos^2\theta = 1$$

This is the most fundamental trigonometric identity. It follows from the Pythagorean theorem:

$$H^2 = P^2 + B^2$$

Dividing throughout by H^2 gives:

$$(P/H)^2 + (B/H)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

6. Derived Pythagorean Identities

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives:

$$1 + \tan^2\theta = \sec^2\theta$$

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ gives:

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

Thus, the three major Pythagorean identities are:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

7. Identities Involving Complementary Angles

If two angles are complementary, their sum is 90°. The following relationships are called co-function identities:

$$\sin(90^\circ - \theta) = \cos\theta$$

$$\cos(90^\circ - \theta) = \sin\theta$$

$$\tan(90^\circ - \theta) = \cot\theta$$

$$\cot(90^\circ - \theta) = \tan\theta$$

$$\sec(90^\circ - \theta) = \mathrm{cosec}\,\theta$$

$$\mathrm{cosec}(90^\circ - \theta) = \sec\theta$$

8. How to Prove a Trigonometric Identity

When proving an identity, simplify one side until it becomes exactly equal to the other side.

1. Start with the more complicated side.
2. Convert secant, cosecant, and cotangent into sine and cosine when useful.
3. Use the fundamental identities $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \mathrm{cosec}^2\theta$.
4. Factor or take a common denominator where appropriate.
5. Do not assume that the left-hand side and right-hand side are equal before proving it.

9. Detailed Solved Examples

Example 1

Prove that $\sin\theta \times \mathrm{cosec}\,\theta = 1$.

Using the reciprocal identity: $\mathrm{cosec}\,\theta = 1/\sin\theta$.

Therefore, $\sin\theta \times \mathrm{cosec}\,\theta = \sin\theta \times (1/\sin\theta)$.

The factors cancel, giving 1.

Answer: $\sin\theta \times \mathrm{cosec}\,\theta = 1$.

Example 2

Prove that $\tan\theta \times \cot\theta = 1$.

Use $\tan\theta = \sin\theta/\cos\theta$ and $\cot\theta = \cos\theta/\sin\theta$.

Therefore, $\tan\theta \times \cot\theta = (\sin\theta/\cos\theta)(\cos\theta/\sin\theta)$.

Cancel $\sin\theta$ and $\cos\theta$.

Answer: $\tan\theta \times \cot\theta = 1$.

Example 3

Prove that $1 + \tan^2\theta = \sec^2\theta$.

Start with the fundamental identity: $\sin^2\theta + \cos^2\theta = 1$.

Divide every term by $\cos^2\theta$.

$\sin^2\theta/\cos^2\theta + \cos^2\theta/\cos^2\theta = 1/\cos^2\theta$.

Using $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$, obtain $\tan^2\theta + 1 = \sec^2\theta$.

Answer: $1 + \tan^2\theta = \sec^2\theta$.

Example 4

Prove that $1 + \cot^2\theta = \text{cosec}^2\theta$.

Start with $\sin^2\theta + \cos^2\theta = 1$.

Divide every term by $\sin^2\theta$.

$\sin^2\theta/\sin^2\theta + \cos^2\theta/\sin^2\theta = 1/\sin^2\theta$.

Using $\cot\theta = \cos\theta/\sin\theta$ and $\text{cosec}\,\theta = 1/\sin\theta$, obtain $1 + \cot^2\theta = \text{cosec}^2\theta$.

Answer: $1 + \cot^2\theta = \text{cosec}^2\theta$.

Example 5

Simplify: $(1 - \sin^2\theta)$.

Use the identity $\sin^2\theta + \cos^2\theta = 1$.

Rearrange it: $1 - \sin^2\theta = \cos^2\theta$.

Answer: $\cos^2\theta$.

Example 6

Simplify: $(1 - \cos^2\theta)$.

From $\sin^2\theta + \cos^2\theta = 1$, subtract $\cos^2\theta$ from both sides.

Therefore, $1 - \cos^2\theta = \sin^2\theta$.

Answer: $\sin^2\theta$.

Example 7

Simplify: $\sec^2\theta - \tan^2\theta$.

Use the identity $1 + \tan^2\theta = \sec^2\theta$.

Rearrange: $\sec^2\theta - \tan^2\theta = 1$.

Answer: 1.

Example 8

Simplify: $cosec^2\theta - cot^2\theta$.

Use the identity $1 + cot^2\theta = cosec^2\theta$.

Rearrange: $cosec^2\theta - cot^2\theta = 1$.

Answer: 1.

Example 9

Prove that $\tan\theta = \sin\theta/\cos\theta$.

Use the definitions $\sin\theta = P/H$ and $\cos\theta = B/H$.

Therefore, $\sin\theta/\cos\theta = (P/H)/(B/H)$.

Dividing by a fraction means multiplying by its reciprocal: $(P/H)(H/B)$.

Cancel H to obtain P/B.

Since $\tan\theta = P/B$, the result follows.

Answer: $\tan\theta = \sin\theta/\cos\theta$.

Example 10

Prove that $\cot\theta = \cos\theta/\sin\theta$.

Using $\cos\theta = B/H$ and $\sin\theta = P/H$.

$\cos\theta/\sin\theta = (B/H)/(P/H)$.

This equals $(B/H)(H/P) = B/P$.

Since $\cot\theta = B/P$, the identity is proved.

Answer: $\cot\theta = \cos\theta/\sin\theta$.

Example 11

If $\sin\theta = 3/5$, find $\cos^2\theta$ using an identity.

Use $\sin^2\theta + \cos^2\theta = 1$.

Substitute $\sin\theta = 3/5$: $(3/5)^2 + \cos^2\theta = 1$.

$9/25 + \cos^2\theta = 1$.

Therefore, $\cos^2\theta = 16/25$.

Answer: $\cos^2\theta = 16/25$.

Example 12

If $\tan\theta = 3/4$, find $\sec^2\theta$.

Use $1 + \tan^2\theta = \sec^2\theta$.

Substitute $\tan\theta = 3/4$.

$\sec^2\theta = 1 + (3/4)^2 = 1 + 9/16 = 25/16$.

Answer: $\sec^2\theta = 25/16$.

Example 13

If $\cot\theta = 5/12$, find $\mathrm{cosec}^2\theta$.

Use $1 + \cot^2\theta = \mathrm{cosec}^2\theta$.

Substitute $\cot\theta = 5/12$.

$\mathrm{cosec}^2\theta = 1 + (5/12)^2 = 1 + 25/144 = 169/144$.

Answer: $\mathrm{cosec}^2\theta = 169/144$.

Example 14

Simplify: $(\sin\theta/\cos\theta) + (\cos\theta/\sin\theta)$.

Take the common denominator $\sin\theta\cos\theta$.

The numerator becomes $\sin^2\theta + \cos^2\theta$.

Therefore, the expression becomes $(\sin^2\theta + \cos^2\theta)/(\sin\theta\cos\theta)$.

Using $\sin^2\theta + \cos^2\theta = 1$, the result is $1/(\sin\theta\cos\theta)$.

Answer: $1/(\sin\theta\cos\theta)$.

Example 15

Prove that $(1 + \tan^2\theta) / \sec^2\theta = 1$.

Use the identity $1 + \tan^2\theta = \sec^2\theta$.

Substitute the numerator with $\sec^2\theta$.

The expression becomes $\sec^2\theta/\sec^2\theta = 1$.

Answer: 1.

Example 16

Prove that $(1 + \cot^2\theta) / \mathrm{cosec}^2\theta = 1$.

Use $1 + \cot^2\theta = \mathrm{cosec}^2\theta$.

Replace the numerator by $\mathrm{cosec}^2\theta$.

The expression becomes $\mathrm{cosec}^2\theta/\mathrm{cosec}^2\theta = 1$.

Answer: 1.

Example 17

Simplify: $(\sec^2\theta - 1)/\tan^2\theta$.

From $1 + \tan^2\theta = \sec^2\theta$, we get $\sec^2\theta - 1 = \tan^2\theta$.

Therefore, the expression becomes $\tan^2\theta/\tan^2\theta$.

Answer: 1.

Example 18

Simplify: $(\mathrm{cosec}^2\theta - 1)/\cot^2\theta$.

From $1 + \cot^2\theta = \mathrm{cosec}^2\theta$, we get $\mathrm{cosec}^2\theta - 1 = \cot^2\theta$.

Therefore, the expression becomes $\cot^2\theta/\cot^2\theta$.

Answer: 1.

Example 19

If $\sin\theta = 5/13$, find $\tan\theta$ using identities.

Use $\sin^2\theta + \cos^2\theta = 1$.

$\cos^2\theta = 1 - (5/13)^2 = 1 - 25/169 = 144/169$.

For an acute angle, $\cos\theta = 12/13$.

Therefore, $\tan\theta = \sin\theta/\cos\theta = (5/13)/(12/13)$.

Answer: 5/12.

Example 20

If $\sec\theta = 13/12$, find $\tan\theta$.

Use $1 + \tan^2\theta = \sec^2\theta$.

$\tan^2\theta = \sec^2\theta - 1$.

$\tan^2\theta = (13/12)^2 - 1 = 169/144 - 144/144 = 25/144$.

For an acute angle, $\tan\theta = 5/12$.

Answer: 5/12.

10. Important Algebraic Techniques

Trigonometric identities often require ordinary algebraic techniques.

• Take a common denominator when fractions have different denominators.

• Factorise expressions before cancelling common factors.

• Replace $\sin^2\theta + \cos^2\theta$ with 1 whenever it appears.

• Replace $\sec^2\theta - \tan^2\theta$ with 1.

• Replace $\operatorname{cosec}^2\theta - \cot^2\theta$ with 1.

• Convert all ratios to sine and cosine when a direct simplification is not obvious.

11. Common Mistakes

• Do not cancel terms across addition or subtraction. For example, $(\sin\theta + \cos\theta)/\sin\theta$ cannot be simplified by cancelling $\sin\theta$ from only one term.

• Do not use an identity in the wrong direction.

• Remember that $\sin^2\theta$ means $(\sin\theta)^2$, not $\sin(2\theta)$.

• When taking square roots, consider the sign of the angle and the quadrant if the problem is not restricted to an acute angle.

• While proving an identity, avoid changing both sides simultaneously unless the method explicitly permits it.

12. Quick Revision Sheet

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

$$\tan\theta = \sin\theta/\cos\theta$$

$$\cot\theta = \cos\theta/\sin\theta$$

$$\mathrm{cosec}\,\theta = 1/\sin\theta$$

$$\sec\theta = 1/\cos\theta$$

$$\cot\theta = 1/\tan\theta$$

13. Final Revision Points

- The identity $\sin^2\theta + \cos^2\theta = 1$ is the foundation of the three Pythagorean identities.
- To derive $1 + \tan^2\theta = \sec^2\theta$, divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$.
- To derive $1 + \cot^2\theta = \mathrm{cosec}^2\theta$, divide $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.
- For proving identities, begin with the more complicated side and simplify it step by step.
- Keep the denominator restrictions in mind whenever a trigonometric ratio is involved.